Wendy Pelletier Cleaves

Promoting Mathematics Accessibility through Multiple Representations JIGSAWS

There have been innumerable times when, attempting to solve a mathematical problem, I have been unsure of how to start, how to proceed, or how to interpret the problem correctly. Other times, when attempting to solve a mathematics problem with a group of colleagues, I would compare strategies with a partner only to suddenly "see" a simple, straightforward approach. I would tell myself, "If only I had thought to look at it that way" or "I cannot believe how easy this problem is now that I see how you approached it." When working with students who have had limited success with mathematics, we want to provide the tools that allow them to look at problems from different perspectives. The ability to examine problems using varied approaches is one of the most important characteristics of good problem solvers. Other characteristics include independence, flexibility in thinking, and determination, and a willingness to take risks.

Using multiple representations in my teaching is a strategy I find helpful in developing independent problem solvers. To me, the concept of multiple representations fits extremely well with the *jigsaw grouping technique*, a pedagogical approach designed around the idea of placing students in different small groups. This article focuses on using a multiple representations jigsaw as a strategy to promote accessibility to mathematics for more students.

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USING MULTIPLE REPRESENTATIONS

By using multiple representations, students are being asked to show the same information in varied ways. Information could be presented in a table of values, and students could translate that same numerical information into a graph or an equation. Information presented in a story problem could then be represented by students using pictures and numbers. Refer to **table 1** for a sample list of multiple representations used often in middle school mathematics.

According to *Principles and Standards for School Mathematics*, “Students can develop and deepen their understanding of mathematical concepts and relationships as they create, compare, and use various representations. Representations—such as physical objects, drawings, charts, graphs, and symbols—also help students communicate their thinking” (NCTM 2000, p. 280). By moving in and out of different representations, students make more connections with multiple mathematical ideas, see the bigger picture of mathematics, and recognize that there is always more than one way to both represent and solve a problem. Research shows that an important aspect of developing resourceful problem solvers is the ability to draw conclusions from multiple mathematical representations (Donovan and Bransford 2005).

Table 1 Sample list of mathematical representations

Type of Representation	Examples
Numerical	Table of values
Graphical	Graph
Pictorial	Picture or diagram
Verbal	Story or description
Symbolic	Equation or expression
Physical	Manipulatives or concrete model

Table 2 Examples of mathematics jigsaw lessons

Expert Group Assignment	Home Group Assignment
Research a particular class of numbers (prime, composite, square, triangular, rational, irrational, and so on)	Classify a set of numbers
Develop an algorithm for a particular integer operation (addition, subtraction, multiplication, division) and defend using examples and manipulatives	Write a pamphlet on integer operations
Collect data using different experiments	Model data using functions
Summarize and explain a section of the current unit	Study for a unit test

WHAT IS A JIGSAW?

A *jigsaw* is a form of flexible grouping in which class participants become part of two different groups—a home group and an expert group (Aronson 1978; Clarke 1999). The jigsaw management strategy is attributed to Elliot Aronson who developed and used this technique in the 1970s with students at both the University of Texas and the University of California.

Students begin a lesson in home groups and are then separated into multiple expert groups. The lesson’s content material is split into pieces. Each expert group will receive one piece of this material that is *different* from that of the other groups, thereby making each group an expert on its respective material. Expert groups complete their work. Then, each student returns to his or her home group to report their findings. This particular step is what makes the jigsaw a cooperative *learning* technique—the students need to listen and interact with one another because every student provides access to material that no one else has. Home group members then work to piece together all the material. Without all the pieces, the puzzle remains incomplete, so it is necessary for group members to work together. To be successful, students must compare, contrast, and analyze information; pose new questions; and learn from one another. When implemented effectively, cooperative learning techniques, such as the jigsaw method, are very likely to produce positive results in student achievement (Johnson, Johnson, and Stanne 2000).

JIGSAW GROUPING STRATEGIES

When working with students who have never been part of a jigsaw, I find it helpful to use colors to name groups. Each expert group will be assigned a different color (red, blue, and so on), and each home group will be referred to as a rainbow, in that it has one representative from each color group. Color groups can be formed randomly by having each rainbow group count off to form the new groups, or teachers can select rainbow group members to join a particular color group. Another approach to grouping students is to use letter or number cards. Each home group can be assigned a letter, and each expert

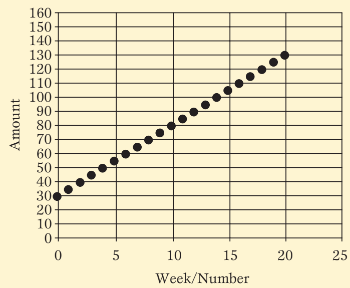
Fig. 1 "Saving for a Rainy Day" jigsaw task cards

Color Group Directions

1. Analyze the information presented, and draw some conclusions.
2. What is the amount of money at the end of week 6?
3. What is the amount of money at the end of week 26?
4. At the end of what week would the student have saved $300?
5. Write an equation to show how the student's savings changed throughout the year.

Green Group Information

Moshon's savings changed during the last year, as described below. The numbers indicate amounts of money (in dollars) at the end of each week. The graph describes Moshon's savings at the end of each of the first 20 weeks. The graph continues in the same way for the rest of the year.



Blue Group Information

Dina's savings changed during the last year, as described below. The numbers indicate amounts of money (in dollars) at the end of each week. The table shows how much money Dina had saved at the end of each week. The table continues in the same way for the rest of the year.

Week	1	2	3	4	5	6	7	8	9	. . .
Amount	7	14	21	28	35	42	49	56	63	. . .

White Group Information

Yonni's savings changed during the last year, as described below. The numbers indicate amounts of money (in dollars) at the end of each week. Yonni kept his savings at $300 throughout the year.

Yellow Group Information

Danny's savings changed during the last year, as described below. The numbers indicate amounts of money (in dollars) at the end of each week. Danny's savings can be described by the expression $300 - 5x$, where x stands for the number of weeks.

Rainbow Group Directions

1. Compare the savings of the four students. Use words like—

   - "the savings increase [or decrease],"
   - "the savings increase or decrease at a rate of . . . ,"
   - "who has a larger [or smaller] amount at the beginning [or end],"
   - "larger [or smaller] by . . . , double . . . , equal. "

   Use tables, graphs, expressions, and explanations.

2. Joe is another student who is saving his money. Create a savings plan for him. His plan should be different from the other plans. Represent the savings plan using a table, a graph, words, and an expression and compare it with the others'.

group, a number. Every student then receives a card with both a letter and a number, identifying both groups to which they belong. In this article, home groups will be referred to as *rainbow groups* and expert groups will be referred to as *color groups*.

MATHEMATICAL JIGSAW LESSONS

Readers may be familiar with the use of the jigsaw in other content areas, such as in history or in language arts, but the jigsaw can also be implemented in the mathematics classroom. Refer to **table 2** for some lesson examples. I have also created and implemented jigsaws that focus on the use of multiple representations. When creating a representations jigsaw, color groups can be given the *same* information presented in different ways or they can receive *different* information presented in different ways. The key is that each color group becomes expert in one form of representation for that specific task. When students rejoin their rainbow group, they share what they learned using their representation: how to pull information from it, how to identify key pieces of information in it, advantages and disadvantages, and so on. The two jigsaw lessons described will give you a sense of how they can be used in your mathematics classroom.

"SAVING FOR A RAINY DAY" JIGSAW

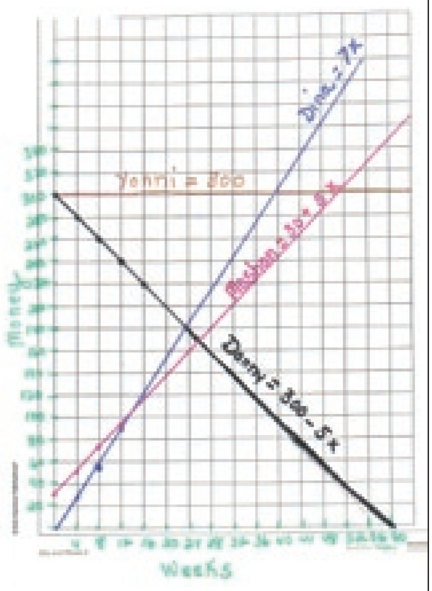
This lesson is adapted from a chapter by Friedlander and Tabach in the NCTM Yearbook *The Roles of Representation in School Mathematics* (NCTM 2001). Refer to **figure 1** for the jigsaw task cards. The "Saving for a Rainy Day" lesson compares different savings plans. The lesson is an example of using a jigsaw to extend students' thinking—moving within and between multiple representations of algebra and discovering how rates of change can be represented in tables, graphs, equations, and story problems.

Fig. 2 A table of values compares all savings plans.

COMPARING 4 PLANS

NAME	START ($)	CHANGE ($/wk)	AMT @ 6 wks (#)	AMT @ 26 wks ($)	When will there be $300	ALG EXTN
DINA	0	+7	42	182	43	7w
DANNY	300	−5	270	170	0	300−5w
YONNI	300	0	300	300	All weeks	300
MOSHON	30	+5	60	210	54	300+5w
JOE	−25	+10	35	235	33	−25+10w

Fig. 3 A graph compares all savings plans.



Students as early as grade 6 can begin to explore positive, negative, and zero slope through this problem.

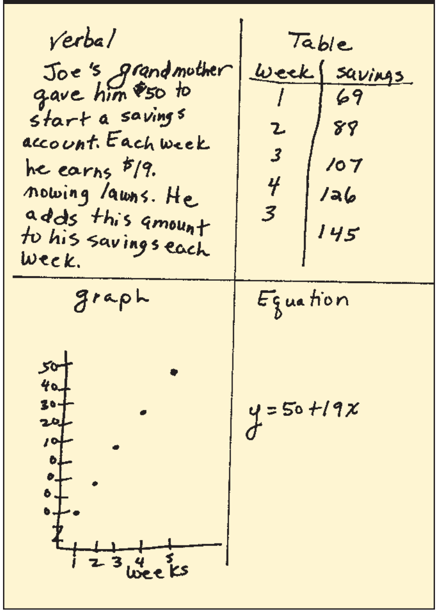
In this jigsaw, each rainbow group will contain a green, a blue, a white, and a yellow group member. Students join their respective color groups and work with information on the savings plan of one of four students—Moshon, Dina, Yonni, or Danny. This information is represented differently for each color group. For the green group, Moshon's savings plan is represented in a graph. The blue group analyzes Dina's plan found in a table of values. The white group will read the written description of Yonni's plan. The yellow group will study Danny's plan as represented with an algebraic expression. Each color group has to respond to the same questions for their given student (see the task cards in **fig. 1** for specific questions). Once these questions are answered, the group then needs to re-represent that student's savings plan using a different representation.

Once they are back in their rainbow groups, students share what happened in each color group. Then they must work together to compare the four savings plans and determine which student is saving the most. To compare and contrast these savings plans, students will realize that they need to select one representation, such as a table of values, and translate all savings plans into that representation. Deciding on which representation to choose forces groups to analyze the advantages and disadvantages of each representation. Refer to **figures 2** and **3** showing how two different groups decided to represent all the savings plans.

Figure 2 shows a group's extensive table of values. Each student's initial amount of money, the amount of change each week, the amount after a certain number of weeks, the time at which $300 is reached, and the algebraic expression for each savings plan are listed. In discussion with this group, I asked them to explain the connection among the initial savings amounts, the changes each week, and the algebraic expressions. They explained that the algebraic expression is found by taking the initial amount and adding on the change per week multiplied by w, the number of weeks. When asked why Moshon's expression did not follow this pattern, they were quick to point out that it should be $30 + 5w$. Research shows that students often misinterpret the values for slope (rate of change) and y-intercept (initial savings amount) in symbolic equations. However, working through problems set in context, such as this lesson, helps to minimize confusion between these two concepts (Donovan and Bransford 2005).

Figure 3 shows how a group of students represented the savings plans in a graph. When I asked this group's members why a graph was chosen, they explained that for them it was the easiest way to compare all four plans at once. They could easily see who was saving the most money and who was spending the most money based on the steepness of the graph. They could also quickly answer questions about how much each student started with or how much money a particular student had for a given week. When asked to describe the downside to using a graph, this group did concede that a graph is more time-consuming to produce and that the answers obtained are not always as accurate as those in a table of values. In fact, the lines representing Dina's and Moshon's savings plans were drawn inaccurately and could lead the observer to make false conclusions.

Fig. 4 One group's representation of a new student's savings plan



In addition, each rainbow group was asked to create a savings plan for Joe, a new student in the list. This step assesses how well the group learned about multiple representations from one another. I found it interesting that the group whose work appears in **figure 2** chose a creative plan for Joe, starting off his savings "in the red" at –$25. Refer to **figure 4** for an example of another group's plan for Joe. This group chose a plan in which his account starts with $50 and has a positive rate of change, earning $19 per week mowing lawns. The linear equation $y = 50 + 19x$ models the behavior in both the table of values and the graph. Aside from missing a y-axis label and title on its graph, this group's members have shown that they understand how to represent the same mathematical information in multiple ways.

"SOLVING SYSTEMS OF EQUATIONS" JIGSAW

This next jigsaw lesson, "Solving Systems of Equations," is adapted from mathematical problem-solving tasks presented in the *Mathematics in Context* unit titled "Comparing Quantities"

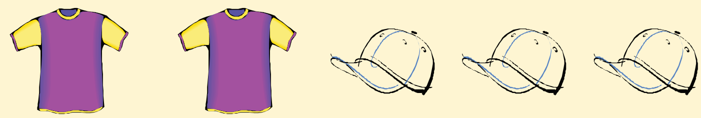
Fig. 5 "Solving Systems of Equations" jigsaw task cards

Color Group Directions
Solve the problem on your task card.

Red Group Information
The Purple Sox Pro-Shop has the following special deals:

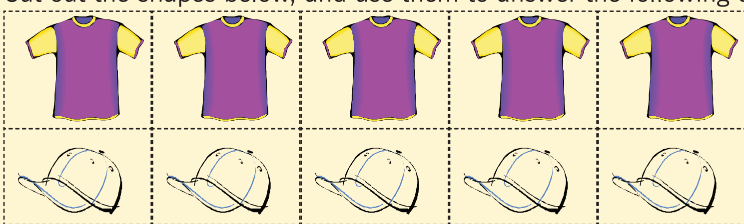
Buy 3 shirts and 2 baseball caps for only $54.50.

Buy 2 shirts and 3 baseball caps for only $50.50.



1. Write a rule for each special deal.
2. What is similar about the two deals above? What is different?
3. How could you find out how much 1 baseball cap costs? How much is 1 shirt?

Orange Group Information
Cut out the shapes below, and use them to answer the following questions:



The Purple Sox Pro-Shop has the following special deals:

- Buy 3 shirts and 2 baseball caps for only $54.50.
- Buy 2 shirts and 3 baseball caps for only $50.50.

1. Write a rule for each special deal.
2. What is similar about the two deals above? What is different?
3. How could you find out how much 1 baseball cap costs? How much is 1 shirt?

Yellow Group Information
At the ball game, Jerry bought 2 shirts and 3 baseball caps. The total cost was $50.50. Tom bought 3 shirts and 2 baseball caps. His purchase total was $54.50.

1. How much does 1 shirt cost?
2. How much does 1 baseball cap cost?

Green Group Information
Solve the following:

$$3x + 2y = 54.50$$
$$2x + 3y = 50.50$$

Fig. 5 "Solving Systems of Equations" jigsaw task cards (continued)

Blue Group Information

Sam works at the Purple Sox Pro-Shop, and he is very busy. He moves from one customer to another, writing down orders for shirts and baseball caps. The table below shows you how he writes the orders on his note pad:

Order	No. of Shirts	No. of Baseball Caps	Total Cost
1	2	3	\$50.50
2	3	2	\$54.50
3	4	6	
4	6	4	
5	9	6	
6	5	0	
7			
8			
9			
10			

1. Some of the orders do not have total costs listed. What are the prices of these orders?
2. Make up two new orders, and write them in the notebook. Fill in the prices of these orders.
3. What is the price of 1 shirt?
4. What is the price of 1 baseball cap?

Purple Group Information

Number of Shirts

5						
4						
3			\$54.50			
2				\$50.50		
1						
0						
	0	1	2	3	4	5

Number of Baseball Caps

1. What does the \$54.50 represent in the chart above?
2. What does the \$50.50 represent in the chart above?
3. How much would 3 baseball caps cost? How much would 3 shirts cost?
4. How did you use the chart to figure out these prices?
5. How much are 5 baseball caps and 4 shirts?
6. How much is 1 baseball cap? How much is 1 shirt?

Rainbow Group Directions

Have each person share what was on his or her color group's task card. Share what the color group discovered about the problem and solution methods. Summarize what similarities and differences you have identified.

(Kindt et al. 2006). Refer to **figure 5** for the jigsaw task cards. This lesson explores the use of nontraditional methods to solve systems of equations with two variables and illustrates how a jigsaw can be used to demonstrate many different approaches to solving the same problem. In this way, more students can gain access to seemingly abstract, symbolic mathematical ideas such as solving systems of linear equations. In fact, most of the solution methods used in this lesson are appropriate for students in grades 6 and above.

In this jigsaw, students work in color groups that each start with a different representation of the same information about ball game sales. The red group deals with illustrations; orange, physical models; yellow, a story problem (verbal representation); green, equations (symbolic representations); blue, notebook notation (numerical representation); and purple, a combination chart (graphical representation). Students are simply asked to solve the problem. The power of this lesson becomes evident when the students rejoin their rainbow group and one by one discover that all color groups started with different representations of the same system of linear equations. Students discuss the method that their color group used to solve the problem, then the rainbow group's task involves solving other systems of equations by applying the methods just presented.

I wanted to share this lesson as an example of a jigsaw in which all color groups are given the same information represented in multiple ways. Implementing this type of jigsaw with students is especially exciting when they realize that all groups began with the same information. This particular lesson also helps to highlight what I consider to be one of the key components of the jigsaw approach: students learning concepts and skills from their peers during group discussions.

Table 3 Possible jigsaw configurations

Class Size	Possible Options for Expert (Color) Groups	Possible Options for Home (Rainbow) Groups
20	A. 4 color tasks—groups of 5 students each B. 5 color tasks—groups of 4 students each	A. 5 groups of 4 students B. 4 groups of 5 students
23	A. 4 color tasks—3 groups of 6 students and 1 group of 5 students B. 5 color tasks—2 groups of 4 students and 3 groups of 5 students	A. 3 groups of 5 students (1 repeat color), and 2 groups of 4 students (1 from each color) B. 3 groups of 6 students (1 repeat color) and 1 group of 5 students (1 from each color)
36	A. 6 color tasks—6 groups of 6 students B. Two mini jigsaws of 18 students each: 6 color tasks—6 groups of 3 students	A. 6 groups of 6 students B. 3 groups of 6 students (two sets—one for each jigsaw)

Through the use of multiple representations, more students can have success with key mathematical ideas that, in the past, may have been off-limits to them because of their abstract nature. By providing students with an opportunity to explore different ways to represent the same information, more students can gain access to algebra even though they may have had limited success with traditional symbolic methods.

BENEFITS TO USING MULTIPLE REPRESENTATION JIGSAWS

The flexible structure of the jigsaw makes this form of grouping effective with different types of content, tasks, and learning styles and ideal for differentiated instruction. To provide access to the same content, consider assigning students to particular color groups and have struggling learners begin with a more concrete representation. By giving English language learners the opportunity to begin with physical models or pictorial representations, you allow them access into the mathematics. Multiple representation jigsaws can also lend themselves to grouping according to varied learning styles (kinesthetic, visual, and verbal-auditory learning) and multiple intelligences. Students can be assigned to color groups in such a way that their task matches their dominant learning style or intelligence. They extend their thinking in rainbow groups where each member represents a different style or level of intelligence.

Jigsaws can be modified to accommodate any number of students in your classroom. At times, you may find that you have more than one member of a color group in any given rainbow group or you may find that you need to run two mini jigsaws simultaneously (refer to **table 3** for examples of different jigsaw configurations).

Using multiple representations in tandem with the jigsaw technique can give students opportunities to gain access to rich mathematics. We know how critical it is to get students involved in their learning; students who are hesitant to participate in whole-class discussions are more likely to engage in small-group work during jigsaw lessons. According to *Principles and Standards for School Mathematics,* middle school mathematics teachers should "strive to establish a communication-rich classroom in which students are encouraged to share their ideas and to seek clarification until they understand" (NCTM 2000, p. 271). The student-to-student communication that occurs within a jigsaw lesson can serve this purpose. Using multiple representations in this way also promotes student independence, leading more students to think: "If I know this representation and I know how to translate from one to another, then I know I can solve this problem!"

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